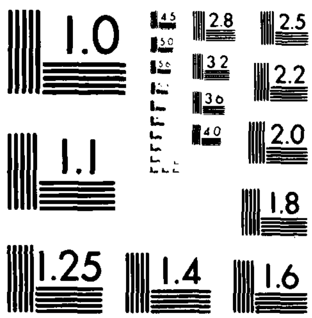
AD-A089 937 BRONX COMMUNITY COLL NY DEPT OF PHYSICS F/G 20/3
TRANSMISSION OF ELECTROMAGNETIC RADIATION THROUGH NON-PLANAR LA--ETC(U)
MAR 80 L A DEACETIS AFOSR-79-0035

UNCLASSIFIED AFOSR-TR-80-0757 NL

END
DATE
FILMED
11 80
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

TRANSMISSION OF ELECTROMAGNETIC RADIATION THROUGH NON-PLANAR LAYERED MEDIA

FINAL

Louis A. DeAcetis

Department of Physics
Bronx Community College
of The City University of New York
Bronx, New York 10453

March 1980

Prepared for

Air Force Office of Scientific Research
Bolling AFB
D.C. 20332

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 80-0757	2. GOVT ACCESSION NO. AD-A089937	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Transmission of Electromagnetic Radiation through Non-Planar Layered Media		5. TYPE OF REPORT & PERIOD COVERED Final
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Louis A. DeAcetis		8. CONTRACT OR GRANT NUMBER(s) AFOSR 79-0035
9. PERFORMING ORGANIZATION NAME AND ADDRESS Physics Department, Bronx Community College, Bronx, New York 10453		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2301/A6
11. CONTROLLING OFFICE NAME AND ADDRESS AFOSR/NP, Bolling AFB, Bldg. 410, Washington, DC 20332		12. REPORT DATE March 1980
		13. NUMBER OF PAGES 43
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release;
distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

BESSEL FUNCTIONS
COMPUTER SUBPROGRAMS
ELECTROMAGNETIC WAVES
LAYERED MEDIA
MATHEMATICAL FUNCTIONS
RADOMES

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An investigation of transmission of electromagnetic radiation through non-planar media consisting of several layers of dielectric has begun. A library search of recent theoretical and experimental work in this area was conducted, and a theoretical study of the exact solutions to transmission through layered cylindrical and spherical media is underway. Computer programs to generate the spherical and half-integer Bessel Functions of both the first and second kind are included in this report.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

ABSTRACT

An investigation of transmission of electromagnetic radiation through non-planar media consisting of several layers of dielectric has begun. A library search of recent theoretical and experimental work in this area was conducted, and a theoretical study of the exact solutions to transmission through layered cylindrical and spherical media is underway. The library survey did not uncover any systemic experimental investigations in this area. Attempts at expanding exact solutions to obtain suitable approximations to the plane-layer cases have thus far not been successful. Several computer programs not generally available were developed to generate the spherical and half-integer Bessel Functions of both the first and second kind and are included in this report. Calculations using these functions are currently underway.

TABLE OF CONTENTS

I. INTRODUCTION

The investigation of transmission of electromagnetic radiation through non-planar layered media is of practical interest in the design and construction of wideband, lensed radomes. Work in this area [1-4] represents an exercise in the transmission of electromagnetic radiation through a layered medium, and most such analyses use the approximation that the radiation penetrates a surface that can be considered as "locally plane."[5-7] This permits the use of the "characteristic matrices"[8] for the stratified medium, and the plane sheet transmission coefficients. Aside from the manner of the approximation is the form, which is especially suitable for calculations on a computer. It was the purpose of this investigation to address the question of the accuracy of the "locally plane" approximation, which has been widely used in the past,[4,7,9-11] and to seek other approximations which might be used to more accurately determine the fields penetrating a non-planar layered medium.

Most discussions of diffraction by a curved dielectric surface [12-19] deal with its scattering characteristics (e.g., backscattering cross section) rather than transmission. Transmission investigations through multilayer structures generally involve planes of dielectrics [20] and these still generally draw on the work of Albeles[21] as discussed in Born and Wolf,[8] Analysis of transmission through curved "electromagnetic

windows" generally fall into three categories: ray-tracing[22]; ray-tracing augmented by the plane-sheet transmission coefficients;[9, 10,23] and "Wave Spectrum" surface integration techniques, including the plane wave spectrum (PWS)[1,2,4,10,11] and the spherical wave spectrum.[24]

Based upon the results of a library search of previous work, it was decided to first investigate transmission through non-planar, layered structures for which exact solutions were possible, viz., concentric cylinders and concentric spheres. These solutions would then lay a foundation for a second stage of investigation which would consist of two parts: a) comparison with results obtained from calculations using the "locally plane" approximation, and b) expansion of the exact solutions in the limit of large radius, hopefully obtaining results which would suggest suitable correction terms which would serve to improve the accuracy of the calculations for curved surfaces.

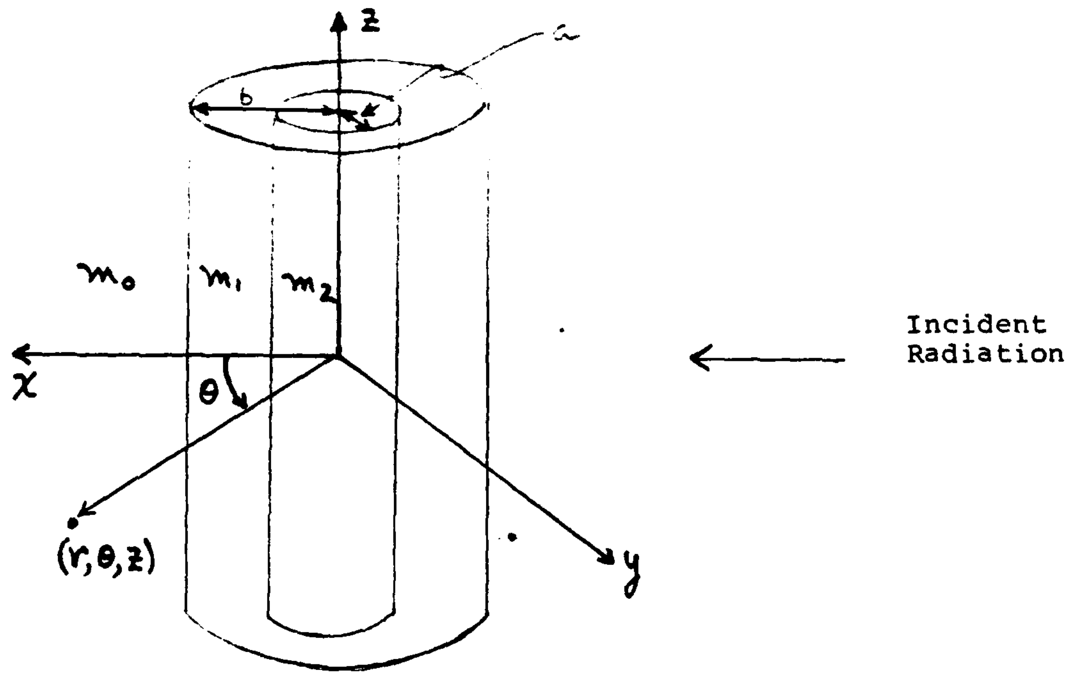
Stage 1 work was readily accomplished by expanding on the work of Kerker, et. al.[25] Work in stage 2, however has not been completed as yet. Attempts to expand the exact solutions in the limit of large radius for both the cylinder and the sphere have thus far not yielded formulations which can be readily implemented in transmission calculations (i.e., appropriate correction "factors" or "terms" which could be combined

with or replace the "characteristic matrix" usually used in such calculations). Stage 2, part (a) calculations have been delayed because of a lack of computer subprograms to calculate values for the half-integer Bessel Functions and the so-called "Riccati-Bessel Functions". The latter programs have been written (in both the BASIC and the FORTRAN IV language) and are included in the appendix of this document. Calculations using these programs are currently underway, but unfortunately have not been completed for inclusion here. It is expected that these results will be given in a future communication and publication.

A summary of the theoretical investigations that have been carried out thus far is given below.

II. Fields Within Concentric Circular Cylinders

Kerker[25] and Matijevic[13] have dealt with the case of scattering by coaxial dielectric cylinders. Whereas they are concerned mainly with fields external to the cylinders (scattering), we will consider the fields within. Consider plane electromagnetic radiation traveling in x-direction, incident upon two coaxial cylinders. Let a=radius of the inner cylinder which has index of refraction m_2; let b=radius of outer cylinder of index m_1; let m_o=index of the space surrounding the cylinders.



All indices of refraction will be taken as real (non-lossy materials). The field vectors can be related to solutions of the scalar wave equation in circular cylindrical coordinates.

$$\nabla^2 u + k^2 u = 0 \tag{1}$$

where we have suppressed the harmonic time dependence factor. This becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0 \tag{2}$$

which is separable for a solution of the form

$$u = R(r)\,\Theta(\theta)\,Z(z). \tag{3}$$

This yields three ordinary differential equations:

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left[\left(k_0^2 - h^2\right)r - n^2\right]R = 0 \tag{4}$$

$$\frac{d^2\Theta}{d\theta^2} + n^2\,\Theta = 0 \tag{5}$$

$$\frac{d^2 Z}{dz^2} + h^2 Z = 0, \tag{6}$$

Where $h=k_o\sin\phi$, with ϕ equal to the angle between direction of propagation of the incident radiation and a normal to the z-axis. Since we are initially assuming the incident radiation is along the x-axis, $\phi=0$ ($\phi\neq0$ for oblique incidence). For the case where the E-vector is incident parallel to the z-axis, the potentials in the three regions are given by:[27]

$$r>b:\ u^{(0)} = \sum_{n=-\infty}^{\infty} F_n\left\{J_n(m_0kr) - b_n^{(0)} H_n(m_0kr)\right\} \tag{7a}$$

$$b>r>a:\ u^{(1)} = \sum_{n=-\infty}^{\infty} F_n\left\{B_n^{(1)} J_n(m_1kr) - b_n^{(1)} H_n(m_1kr)\right\} \tag{7b}$$

$$r<a \qquad u^{(2)} = \sum_{n=-\infty}^{\infty} F_n \left\{ B_n^{(2)} J_n(m_2 kr) \right\} \tag{7c}$$

For the H-vector parallel to the z-axis, we have

$$r>b: \quad v^{(0)} = \sum_{n=-\infty}^{\infty} F_n \left\{ J_n(m_0 kr) - a_n^{(0)} H_n(m_0 kr) \right\} \tag{8a}$$

$$b>r>a: \quad v^{(1)} = \sum_{n=-\infty}^{\infty} F_n \left\{ A_n^{(1)} J_n(m_1 kr) - a_n^{(1)} H_n(m_1 kr) \right\} \tag{8b}$$

$$r<a: \quad v^{(2)} = \sum_{n=-\infty}^{\infty} F_n \left\{ A_n^{(2)} J_n(m_2 kr) \right\} \tag{8c}$$

where

$F_n = (-1)^n e^{in\theta}$

$J_n(x)$=Bessel Functions of the first kind

$Y_n(x)$=Bessel Function of the second kind.

$H_n(x) = J_n(x) - iY_n(x)$=Hankel function of the second kind

and the $A_n^{(0)}$, $A_n^{(1)}$, $A_n^{(1)}$, $A_n^{(2)}$, $B_n^{(1)}$, $B_n^{(2)}$ are coefficients whose values are to be determined from the boundary conditions. Applying the boundary conditions at the interfaces, viz., mu, $m\,\partial u/\partial r$, $m^2 v$, $\partial v/\partial r$ all continuous at the boundaries, we get eight equations:

$$b_n^{(0)} m_0 H_n(m_0\alpha_1) - b_n^{(1)} m_1 H_n(m_1\alpha_1) + B_n^{(1)} m_1 J_n(m_1\alpha_1) = m_0 J_n(m_0\alpha_1) \quad (9a)$$

$$b_n^{(0)} m_0^2 H_n'(m_0\alpha_1) - b_n^{(1)} m_1^2 H_n'(m_1\alpha_1) + B_n^{(1)} m_1^2 J_n'(m_1\alpha_1) = m_0^2 J_n'(m_0\alpha_1) \quad (9b)$$

$$-b_n^{(1)} m_1 H_n(m_1\alpha_2) + B_n^{(1)} m_1 J_n(m_1\alpha_2) - B_n^{(2)} m_2 J_n(m_2\alpha_2) = 0 \quad (9c)$$

$$-b_n^{(1)} m_1^2 H_n'(m_1\alpha_2) + B_n^{(1)} m_1^2 J_n'(m_1\alpha_2) - B_n^{(2)} m_2^2 J_n'(m_2\alpha_2) = 0 \quad (9d)$$

$$a_n^{(0)} m_0^2 H_n(m_0\alpha_1) - a_n^{(1)} m_1^2 H_n(m_1\alpha_1) + A_n^{(1)} m_1^2 J_n(m_1\alpha_1) = m_0^2 J_n(m_0\alpha_1) \quad (9e)$$

$$a_n^{(0)} m_0 H_n'(m_0\alpha_1) - a_n^{(1)} m_1 H_n'(m_1\alpha_1) + A_n^{(1)} m_1 J_n'(m_1\alpha_1) = m_0 J_n'(m_0\alpha_1) \quad (9f)$$

$$-a_n^{(1)} m_1^2 H_n(m_1\alpha_2) + A_n^{(1)} m_1^2 J_n(m_1\alpha_2) - A_n^{(2)} m_2^2 J_n(m_2\alpha_2) = 0 \quad (9g)$$

$$-a_n^{(1)} m_1 H_n'(m_1\alpha_2) + A_n^{(1)} m_1 J_n'(m_1\alpha_2) - A_n^{(2)} m_2 J_n'(m_2\alpha_2) = 0 \quad (9h)$$

$$\alpha_1 = kb$$
$$\alpha_2 = ka$$

The incident field can be expanded in terms of the J_n to yield:

$$u^i = \sum_{n=-\infty}^{\infty} (-i)^n J_n(k_0 r) e^{in\theta} \quad (10)$$

where k_0 = wave number in the region outside the cylinders.

Since we will be interested in the fields that penetrate into the core cylinder, we must evaluate the $A_n^{(2)}$, $B_n^{(2)}$. Using Eqs. (9e-9h), we can write:

$$A_n^{(2)} = \frac{\begin{vmatrix} m_0^2 H_n(m_0\alpha_1) & -m_1^2 H_n(m_1\alpha_1) & m_1^2 J_n(m_1\alpha_1) & m_0^2 J_n(m_0\alpha_1) \\ m_0 H_n'(m_0\alpha_1) & -m_1 H_n'(m_1\alpha_1) & m_1 J_n'(m_1\alpha_1) & m_0 J_n'(m_0\alpha_1) \\ 0 & -m_1^2 H_n(m_1\alpha_2) & m_1^2 J_n(m_1\alpha_2) & 0 \\ 0 & -m_1 H_n'(m_1\alpha_2) & m_1 J_n'(m_1\alpha_2) & 0 \end{vmatrix}}{\begin{vmatrix} m_0^2 H_n(m_0\alpha_1) & -m_1^2 H_n(m_1\alpha_1) & m_1^2 J_n(m_1\alpha_1) & 0 \\ m_0 H_n'(m_0\alpha_1) & -m_1 H_n'(m_1\alpha_1) & m_1 J_n'(m_1\alpha_1) & 0 \\ 0 & -m_1^2 H_n(m_1\alpha_2) & m_1^2 J_n(m_1\alpha_2) & -m_2^2 J_n(m_2\alpha_2) \\ 0 & -m_1 H_n'(m_1\alpha_2) & m_1 J_n'(m_1\alpha_2) & -m_2 J_n'(m_2\alpha_2) \end{vmatrix}} \tag{11}$$

and,

$$B_n^{(2)} = \frac{\begin{vmatrix} m_0 H_n(m_0\alpha_1) & -m_1 H_n(m_1\alpha_1) & m_1 J_n(m_1\alpha_1) & m_0 J_n(m_0\alpha_1) \\ m_0^2 H_n'(m_0\alpha_1) & -m_1^2 H_n'(m_1\alpha_1) & m_1^2 J_n'(m_1\alpha_1) & m_0^2 J_n'(m_0\alpha_1) \\ 0 & -m_1 H_n(m_1\alpha_2) & m_1 J_n(m_1\alpha_2) & 0 \\ 0 & -m_1^2 H_n'(m_1\alpha_2) & m_1^2 J_n'(m_1\alpha_2) & 0 \end{vmatrix}}{\begin{vmatrix} m_0 H_n(m_0\alpha_1) & -m_1 H_n(m_1\alpha_1) & m_1 J_n(m_1\alpha_1) & 0 \\ m_0^2 H_n'(m_0\alpha_1) & -m_1^2 H_n'(m_1\alpha_1) & m_1^2 J_n'(m_1\alpha_1) & 0 \\ 0 & -m_1 H_n(m_1\alpha_2) & m_1 J_n(m_1\alpha_2) & -m_2 J_n(m_2\alpha_2) \\ 0 & -m_1^2 H_n'(m_1\alpha_2) & m_1^2 J_n'(m_1\alpha_2) & -m_2^2 J_n'(m_2\alpha_2) \end{vmatrix}} \tag{12}$$

where we have used $\alpha_1 = k_0 b$
$\alpha_2 = k_0 a$ (13)

with k_0 equal to the wave number of the radiation in the region $r > b$. Eqs. 7c and 8c can then be evaluated to yield the plane polarized components of the incident beam. An arbitrary elliptically polarized wave may be formed by linear superposition. The extention to oblique incidence is made by retaining the "h" term in Eqs. 4 and 6.

III - Fields Within Concentric Spheres

Using the notation of Kerker,[25] the solution for diffraction by concentric dielectric spheres is given in terms of "Hertz - Debye potentials," π_1 and π_2, which satisfy the wave equation in spherical coordinates[28]

$$\frac{1}{r}\left(\frac{\partial^2(r\pi)}{\partial r^2}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\pi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\pi}{\partial\phi^2} + k^2\pi = 0 \quad (14)$$

Let $\pi = R(r)\ \Theta(\theta)\ \Phi(\phi).$ (15)

By separation of variables, this yields the three ordinary differential equations:

$$\frac{d^2(rR)}{dr^2} + \left[k^2 - \frac{n(n+1)}{r^2}\right] rR = 0 \quad (16)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0 \tag{17}$$

$$\frac{d^2\Phi}{d\phi^2}+m^2\,\Phi=0 \tag{18}$$

where n is an integer, and $m=-n,\ldots,o,\ldots,+n$.

The solutions of the radial equation (16) are Riccati-Bessel functions:

$$\psi_n(kr)=\sqrt{\frac{\pi kr}{2}}\;J_{n+\frac{1}{2}}(kr) \tag{19}$$

$$\chi_n(kr)=-\sqrt{\frac{\pi kr}{2}}\;Y_{n+\frac{1}{2}}(kr) \tag{20}$$

where the $J_{n+\frac{1}{2}}$ and $Y_{n+\frac{1}{2}}$ are the half-integer Bessel Functions of the first and second kind. Also,

$$\zeta=\psi_n+i\,\chi_n=\sqrt{\frac{\pi kr}{2}}\;H^{(2)}_{n+\frac{1}{2}}(kr)\ .$$

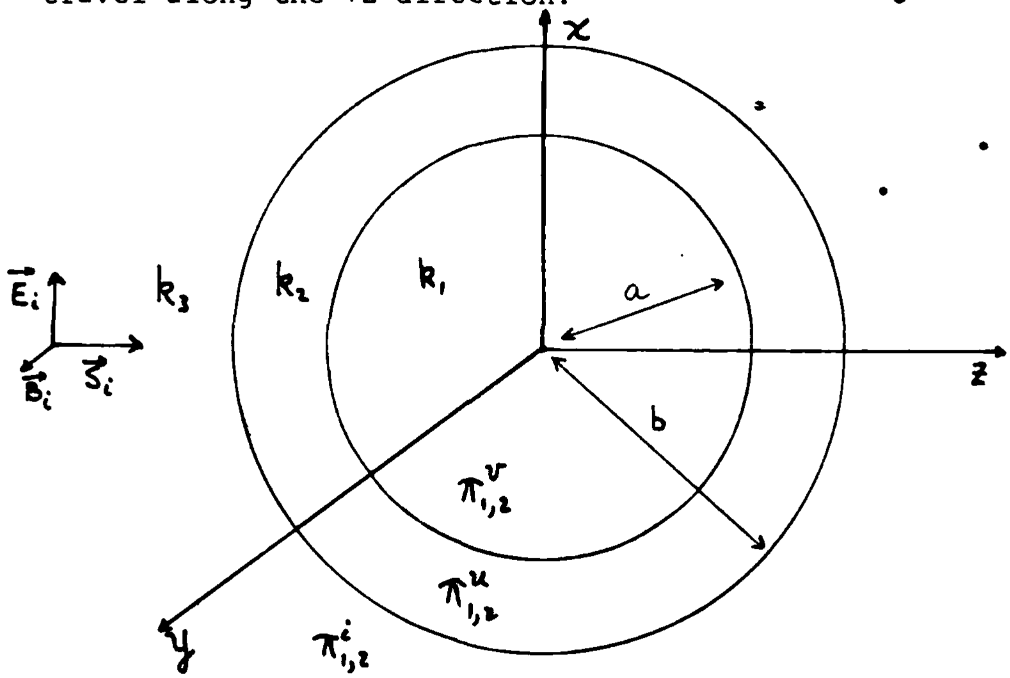
where $H^{(2)}_{n+\frac{1}{2}}(kr)$ =Hankel functions of the second kind.

The solutions to Eq. 17 are the associated Lengendre polynomials,

$$\Theta=P_n^{(m)}(\cos\theta) \tag{21}$$

The solutions of the Φ equation are the harmonic functions of $\sin(m\,\phi)$, $\cos(m\,\phi)$. The general solution to Eq. 14 is a linear superposition of all of the particular solutions multiplied by appropriate coefficients whose values are to be determined from the boundary conditions.

Consider plane radiation incident upon a sphere comprised of two dielectrics. In this case, let the incident radiation travel along the +z direction.



The appropriate Debye potentials for the incident field are of the form:[29]

$$r\,\Pi_1^{i} = \frac{1}{k_3^2}\sum_{n=1}^{\infty} i^{\,n-1}\,\frac{2n+1}{n(n+1)}\,\psi_n(k_3 r)\,P_n^{(1)}(\cos\theta)\,\cos\phi \qquad (22a)$$

$$r\pi_2^i = \frac{1}{k_3^2}\sum_{n=1}^{\infty} i^{n-1}\frac{2n+1}{n(n+1)}\psi_n(k_3r)P_n^{(1)}(\cos\theta)\sin\phi \qquad 22b$$

where we have taken the incident radiation as plane-polarized along the z-axis. The waves in the three regions are obtained from the potentials:

$r>b$:

$$r\pi_1 = -\frac{1}{k_3^2}\sum_{n=1}^{\infty} i^{n-1}\frac{2n+1}{n(n+1)}a_n\zeta_n(k_3r)P_n^{(1)}(\cos\theta)\cos\phi \qquad (23a)$$

$$r\pi_2 = -\frac{1}{k_3^2}\sum_{n=1}^{\infty} i^{n-1}\frac{2n+1}{n(n+1)}b_n\zeta_n(k_3r)P_n^{(1)}(\cos\theta)\sin\phi \qquad (23b)$$

$a<r<b$:

$$r\pi_1^u = -\frac{1}{k_2^2}\sum_{n=1}^{\infty} i^{n-1}\frac{2n+1}{n(n+1)}\left[c_n\psi_n(k_2r)+d_n\chi_n(k_2r)\right]P_n^{(1)}(\cos\theta)\cos\phi \qquad (24a)$$

$$r\pi_2^u = -\frac{1}{k_2^2}\sum_{n=1}^{\infty} i^{n-1}\frac{2n+1}{n(n+1)}\left[e_n\psi_n(kr)+f_n(\chi_n(k_2r)\right]P_n^{(1)}(\cos\theta)\sin\phi \qquad (24b)$$

$r<a$:

$$r\pi_1^v = -\frac{1}{k_1^2}\sum_{n=1}^{\infty} i^{n-1}\frac{2n+1}{n(n+1)}g_n\psi_n(k_1r)P_n^{(1)}(\cos\theta)\cos\phi \qquad (25a)$$

$$r\pi_2^v = -\frac{1}{k_1^2}\sum_{n=1}^{\infty} i^{n-1}\frac{2n+1}{n(n+1)}h_n\psi_n(k_1r)P_n^{(1)}(\cos\theta)\sin\phi \qquad (25b)$$

We impose the boundary conditions at the surfaces r=a and r=b which yield a set of eight equations:

$$m_1 c_n \psi_n'(m_2\alpha) + m_1 d_n \chi_n'(m_2\alpha) + m_2 g_n \psi_n'(m_1\alpha) = 0 \tag{26a}$$

$$c_n \psi_n(m_2\alpha) + d_n \chi_n(m_2\alpha) + g_n \psi_n(m_1\alpha) = 0 \tag{26b}$$

$$c_n \psi_n'(m_2\nu) + d_n \chi_n'(m_2\nu) + m_2 a_n \zeta_n'(\nu) = m_2 \psi_n'(\nu) \tag{26c}$$

$$c_n \psi_n(m_2\nu) + d_n \chi_n(m_2\nu) + a_n \zeta(\nu) = \psi_n(\nu) \tag{26d}$$

$$m_1 e_n \psi_n'(m_2\alpha) + m_1 f_n \chi_n'(m_2\alpha) + m_2 h_n \psi_n'(m_1\alpha) = 0 \tag{26e}$$

$$m_1^2 e_n \psi_n(m_2\alpha) + m_1^2 f_n \chi_n(m_2\alpha) + m_2^2 h_n \psi_n(m_1\alpha) = 0 \tag{26f}$$

$$e_n \psi_n'(m_2\nu) + f_n \chi_n'(m_2\nu) + m_2 b_n \zeta_n'(\nu) = m_2 \psi_n'(\nu) \tag{26g}$$

$$e_n \psi_n(m_2\nu) + f_n \chi_n(m_2\nu) + m_2^2 b_n \zeta_n(\nu) = m_2^2 \psi_n(\nu) \tag{26h}$$

where

$\nu = k_3 b = 2\pi b/\lambda_3$

$\alpha = k_3 a = 2\pi a/\lambda_3$

$m_1 = n_1/n_3$ = relative index of refraction of core

$m_2 = n_2/n_3$ = relative index of refraction of layer

λ_3 = wavelength in outer medium

Since we are interested in the fields in the core, we need the coefficients g_n, h_n. These can be obtained from Eqs. 2 :

$$g_n = \frac{\begin{vmatrix} m_1\psi_n'(m_2\alpha) & m_1\chi_n'(m_2\alpha) & 0 & 0 \\ m_2\psi_n(m_2\alpha) & m_2\chi_n'(m_2\alpha) & 0 & 0 \\ \psi_n'(m_2\nu) & \chi_n'(m_2\nu) & \psi_n'(\nu) & \zeta_n'(\nu) \\ m_2\psi_n(m_2\nu) & m_2\chi_n(m_2\nu) & \psi_n(\nu) & \zeta_n(\nu) \end{vmatrix}}{\begin{vmatrix} \psi_n'(m_2\alpha) & \chi_n'(m_2\alpha) & \psi_n'(m_1\alpha) & 0 \\ m_2\psi_n(m_2\alpha) & m_2\chi_n(m_2\alpha) & m_1\psi_n(m_1\alpha) & 0 \\ \psi_n'(m_2\nu) & \chi_n'(m_2\nu) & 0 & \zeta_n'(\nu) \\ m_2\psi_n(m_2\nu) & m_2\chi_n(m_2\nu) & 0 & \zeta_n(\nu) \end{vmatrix}} \tag{27}$$

$$h_n = \frac{\begin{vmatrix} m_2\psi_n'(m_2\alpha) & m_2\chi_n'(m_2\alpha) & 0 & 0 \\ m_1^2\psi_n(m_2\alpha) & m_1^2\chi_n(m_2\alpha) & 0 & 0 \\ m_2\psi_n'(m_2\nu) & m_2\chi_n'(m_2\nu) & \psi_n'(\nu) & \zeta_n'(\nu) \\ \psi_n(m_2\nu) & \chi_n(m_2\nu) & \psi_n(\nu) & \zeta_n(\nu) \end{vmatrix}}{\begin{vmatrix} m_2\psi_n'(m_2\alpha) & m_2\chi_n'(m_2\alpha) & m_1\psi_n'(m_1\alpha) & 0 \\ \psi_n(m_2\alpha) & \chi_n(m_2\alpha) & \psi_n(m_1\alpha) & 0 \\ m_2\psi_n'(m_2\nu) & m_2\chi_n'(m_2\nu) & 0 & \zeta_n'(\nu) \\ \psi_n(m_2\nu) & \chi_n(m_2\nu) & 0 & \zeta_n(\nu) \end{vmatrix}} \tag{28}$$

The coefficients in Eqs. 25 are thus determined and the exact fields in the core can be evaluated using the spherical coordinate expressions:[30]

$$E_r = \frac{\partial^2(r\pi_1)}{\partial r^2} + k^2 r \cdot \pi_1 \tag{29a}$$

$$E_\theta = \frac{1}{r}\frac{\partial^2(r\pi_1)}{\partial r\,\partial\theta} + i\omega\frac{1}{r\sin\theta}\frac{\partial(r\pi_2)}{\partial\phi} \tag{29b}$$

$$E_\phi = \frac{1}{r\sin\theta}\frac{\partial^2(r\pi_1)}{\partial r\,\partial\phi} - i\omega\frac{1}{r}\frac{\partial(r\pi_2)}{\partial\theta} \tag{29c}$$

$$H_r = \frac{\partial^2(r\pi_2)}{\partial r^2} + k^2 r\pi_2 \tag{29d}$$

$$H_\theta = -i\omega\epsilon\frac{1}{r\sin\theta}\frac{\partial(r\pi_1)}{\partial\phi} + \frac{1}{r}\frac{\partial^2(r\pi_2)}{\partial r\,\partial\theta} \tag{29e}$$

$$H_\phi = i\omega\epsilon\frac{\partial(r\pi_1)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial^2(r^2\pi_2)}{\partial r\,\partial\phi} \tag{29f}$$

IV. Asymptotic Expansions

For both the cylindrical and spherical cases described above, asymptotic expansions of the functions have thus far not proved fruitful (the algebra and manipulation of the determinants is rather unwieldly). The approximations used were that $b-a \ll a$ and a very large. This permits the use of the asymptotic forms of the functions and the Taylor expansion of the derivatives. This approach has been temporarily suspended and will be pursued at a later time.

V. Calculations Using the Exact Solution for Concentric Spheres

This evaluation was attempted first because it was felt it would offer the severest test of the "locally plane" approximation since it is three-dimensional. Computer programs to evaluate the Riccati-Bessel functions (or the spherical Bessel Functions of the first and second kinds, J_n, Y_n, to which they are related by

$$\psi_n = r j_n \quad , \quad \chi_n = r y_n$$

or the half-integer Bessel Functions, $J_{n+\frac{1}{2}}$, $Y_{n+\frac{1}{2}}$ to which the ψ_n, χ_n are related by Eqs. 19 and 20) were not available. It was thus necessary to write these programs, using several recursion techniques described in the literature.[31-33] Documented versions of these programs written in FORTRAN IV and BASIC are included

in the appendix. Calculations using these programs and equations 27-29 are currently underway. These results will be compared with values obtained using the "locally plane" approximation. As originally proposed, similar calculations will be done for the concentric cylinder case.

IV. CONCLUSIONS

A library search of materials accessible to me indicates that, thus far, there has been no systematic investigation (neither theoretical nor experimental) of the "locally plane" approximation to transmission of electromagnetic radiation through non-planar layered media. A theoretical investigation of the transmission into layered cylinders and spheres was done, and calculations based upon these results are underway. Computer programs in FORTRAN IV and BASIC have been written to generate Riccati-Bessel Functions (and spherical and half-integer Bessel-Functions) for use in these calculations. The investigation thus far has given this author some indication of why such investigations have apparently not been done before.

It is expected that this work will continue through the summer (a recent collaborator of the author has expressed some interest in joining the investigation), and further results should be available for submission for publication in the Fall 1980. It is expected that these results will give an indication of the theoretical accuracy of the locally plane approximation, and suggest an appropriate experimental study. Improved approximations to the locally plane approximation are still the ultimate goal.

REFERENCES

1. Groutage, F.D. and D. Smith, "Multi-Layered Tangent Power Series Radome With Integrated Lens," Proceedings of the Fourteenth Symposium on Electromagnetic Windows, J.N. Harris, Editor, Georgia Institute of Technology, June 1978.

2. Huddleston, G.K., H.L. Basset, and J.M. Newton, "Parametric Investigation of Radome Analysis Methods," Proceedings of the Fourteenth Symposium on Electromagnetic Windows, J.N. Harris, Editor, Georgia Institute of Technology, June 1978.

3. Weckesser, L.B., R.K. Frazer, D.J.Yost, B.E. Kuehne, G.P. Tricoles, R. Hayward, and E.L.Rope, "Aerodynamic Heating Effects on Radome Boresignt Errors," Proceedings of the Fourteenth Symposium on Electromagnetic Windows, J.N. Harris, Editor, Georgia Institute of Technology, June 1978.

4. Hayward, R.A., E.L. Rope, and G. Tricoles, "Accuracy of Two Methods for Numerical Analysis of Radome Electromagnetic Effects," Proceedings of the Fourteenth Symposium on Electromagnetic Windows, J.N. Harris, Editor, Georgia Institute of Technology, June 1978.

5. Ref. 2, P. 22.

6. Ref. 3, P. 54.

7. Wu, D.C.F., and R.C. Rudduck, Final Report 2969-4, Contract No. N00019-70-C-0252 (Naval Air Systems Command), Electro-Science Laboratory, Ohio State University, March 1971.

8. Born, M. and E. Wolf, Principles of Optics, Pergammon Press, New York, 1959, p. 50 ff.

9. Tricoles, G., "Radiation Patterns and Boresight Error of a Microwave Antenna Enclosed in an Axially Symmetrical Dielectric Shell," J. Opt. Soc. Am. 54, 1094-1101 (1964).

10. Wu, D.C.F., and R.C. Rudduck, "Plane Wave Spectrum Integration Technique for Radome Analysis," IEEE Trans. Ant. Prop. AP-22, 497-500 (1974).

11. Rudduck, R.C., D.C.F. Wu, and M.R. Intihar, "Near-Field Analysis by the Plane-Wave Spectrum Approach," IEEE Trans-Ant. Prop. AP-21, 231-234 (1973).

12. Adey, A., "Scattering of Electromagnetic Waves by Coaxial Cylinders," Canad. J. Phys. 34, 510-520 (1956).

13. Kerker, M. and E. Matijevic, "Scattering of Electromagnetic Waves from Concentric Infinite Cylinders," J. Opt. Soc. Am. 51, 506-508 (1961).

14. Evans, L. B., J. C. Chen, and S. W. Churchill,"Scattering of Electromagnetic Radiation by Infinitely Long, Hollow, and Coated Cylinders," J. Opt. Soc. Am. 54, 1004-1007 (1964).

15. Farone, W. A., and C. W. Querfeld," Electromagnetic Scattering from Radially Inhomogeneous Infinite Cylinders at Oblique Incidence," J. Opt. Soc. Am. 56, 476-480 (1966).

16. Cohen, A., and P. Alpert," Cross-Sections for Extinction of Tilted Infinite Circular Cylinders," Appl. Opt. 18, 2728 (1979).

17. Yeh, C., "The Diffraction of Waves by an Elliptical Dielectric Cylinder," Technical Report AFCRL-62-151, AD#275539 (1962).

18. Morgan, M. A., S. Chang, K.K. Mei, "Coupled Azimuthal Potentials for Electromagnetic Field Problems in Inhomogeneous Axially Symmetric Media," IEEE Trans. Ant. and Prop. AP-25, 413-417 (1977).

19. Fikioris, J. G., and N. K. Uzunoglu, "Scattering From an Eccentrically Stratified Dielectric Sphere," J. Opt, Soc. Am. 69, 1359-1366 (1979).

20. For a recent example, see A. J. Abuel-Haija, P. J. McMarr, and A. H. Madjid," Computation of the Optical properties of Nonideal Multilayer Structures," App. Opt 18, 3123-3126 (1979). There are many others.

21. Abeles, F., Ann. Phys. Paris 5, 596 (1950).

22. Nagel, F. E., "Ray Transmission Through a Curved Window," App. Opt 8, 1203-1206 (1969).

23. Groutage, F. D., "Radome Development For a Broadband RF Missile Sensor," NELC Technical Report 2023, Naval Air Systems Command, January 1977.

24. Ludwig, A. C., "Near-Field Transformations Using Spherical Wave Expansions," IEEE Trans, Ant. Prop. AP-19, 214-220 (1971).

25. Kerker, M. The Scattering of Light, Academic Press, New York, 1969, pp. 255 ff.

26. Ibid., P. 42

27. Ref. 13, P. 506.

28. Ref. 24, P. 43.

29. Ref. 24, P. 190.

30. Ref. 24, P.41

31. Stegun, I. A., and M. Abramowitz, "Generation of Bessel Functions on High Speed Computers," Mathematical Tables and Other Aids to Computation 11, 255-257 (1957).

32. Goldstein, M. and R. M. Thaler, "Recurrence Techniques for the Calculation of Bessel Functions," Mathematical Tables and Other Aids to Computation 13, 102-108 (1959).

33. Abramowitz, M. and I. A. Stegun, Handbook of Mathematical Functions, Dover Publications, New York, 1965 , Chapter 10.

APPENDIX CONTENTS

```
      SUBROUTINE RBES1(X,N,DELTA,RES,IERR)
C*
C*..RBES1: SUBPROGRAM TO CALCULATE THE RICCATI-BESSEL FUNCTION OF THE 1st KIND.
C*  THIS PROGRAM USES THE SUBPROGRAM BESSP1.
C*
C*..Reference: Stegun and Abramowitz, Eq. 10.3.1.
C*  (See SUBROUTINE BESSP1 for complete references. Note also the comments there
C*  re the recursive nature of the calculations and how to save computing time.)
C*
C*..Dr. Louis A. DeAcetis, February 1980.
C*
C*-------------------
C*
C***Form of the CALL statement:
C*
C*     CALL RBES1(X,N,DELTA,RES,IERR)
C*
C*  Input values:
C*     X = value of the argument; range:           0 <= X <= 100
C*         (This must be a DOUBLE PRECISION value.)
C*     N = order; must be an integer in the range: 0 <= N <= 100
C*     DELTA = Desired accuracy expressed as the fractional error.
C*
C*  Output values:
C*     RES = results (value of the Riccati-Bessel function)
C*           (This is a DOUBLE PRECISION quantity)
C*           The mantissas of values of RES with magnitudes of
C*           less than 1.D-70 have no significance.
C*     IERR = error code: 0 if no error
C*                        1 if X, N, or DELTA out of range (see above).
C*                        2 if accuracy (DELTA) not achieved or in doubt.
C*-------------------
      DOUBLE PRECISION X,RES
C...CASE FOR X=0, ---> RES=0.
      IERR=0
      IF(X.NE.0.DO) GO TO 100
      RES=0.DO
      RETURN
C...HANDLE CASE OF N=0 HERE. USE BESSP1 FOR OTHER CASES:
  100 IF(N.NE.0) GO TO 110
      RES=DSIN(X)
      RETURN
  110 CALL BESSP1(X,N,DELTA,RES,IERR)
      RES=X*RES
      RETURN
      END
C*<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>
```

```
C*
      SUBROUTINE RBES2(X,N,RES,IERR)
C*
C*..RBES2: SUBPROGRAM TO CALCULATE THE RICCATI-BESSEL FUNCTION
C*  OF THE 2nd KIND. THIS PROGRAM USES THE SUBPROGRAM BESSP2.
C*
C*..Reference: Stegun and Abramowitz, Eq. 10.3.1.
C*  (See SUBROUTINE BESSP2 for complete references. Note also the comments there
C*  re the recursive nature of the calculations and how to save computing time.)
C*
C*..Dr. Louis A. DeAcetis, February 1980.
C*
C*-------------------
C*
C***Form of the CALL statement:
C*
C*     CALL RBES2(X,N,RES,IERR)
C*
C*  Input values:
C*    X = value of the argument; range:          0 <= X <= 100
C*        (This must be a DOUBLE PRECISION value.)
C*    N = order; must be an integer in the range: 0 <= N <= 100
C*
C*  Output values:
C*    RES = results (value of the Riccati-Bessel function)
C*          (This is a DOUBLE PRECISION quantity. Tests indicate
C*          results are accurate to better than 7 digits, except
C*          for values > 1.D70 in magnitude.)
C*    IERR = error code: 0 if no error
C*                       1 if X or N out of range (see above).
C*-------------------
      DOUBLE PRECISION X,RES
C...CASE FOR X=0, ---> RES=-1 if N=0; RES= -infinity if N>0
      IERR=0
      IF(X.NE.0.D0) GO TO 100
      RES=-1.D0
      IF(N.EQ.0) RETURN
      RES=-1.D70
      RETURN
C...HANDLE CASE OF N=0, X>0 HERE. GO TO BESSP2 FOR OTHER CASES:
  100 IF(N.NE.0) GO TO 110
      RES=-DCOS(X)
      RETURN
  110 CALL BESSP2(X,N,RES,IERR)
      RES=X*RES
      RETURN
      END
C*<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>
```

```
C*
      SUBROUTINE RBES1P(X,N,DELTA,RES,IERR)
C*
C*..RBES1P: SUBPROGRAM TO CALCULATE THE DERIVATIVE OF THE RICCATI-BESSEL
C*  FUNCTION OF THE 1st KIND.
C*  THIS PROGRAM USES THE SUBPROGRAM BESSP1.
C*
C*..Reference: Stegun and Abramowitz, Eqs. 10.3.1, 10.1.21.
C*  (See SUBROUTINE BESSP1 for complete references. Note also the comments there
C*  re the recursive nature of the calculations and how to save computing time.)
C*
C*..Dr. Louis A. DeAcetis, February 1980.
C*
C*------------------
C*
C***Form of the CALL statement:
C*
C*    CALL RBES1P(X,N,DELTA,RES,IERR)
C*
C*  Input values:
C*    X = value of the argument; range:          0 <= X <= 100
C*        (This must be a DOUBLE PRECISION value.)
C*    N = order; must be an integer in the range: 0 <= N <= 100
C*    DELTA = Desired accuracy expressed as the fractional error.
C*
C*  Output values:
C*    RES = results (derivative of the Riccati-Bessel function)
C*          (This is a DOUBLE PRECISION quantity)
C*          The mantissas of values of RES with magnitudes of
C*          less than 1.D-70 have no significance.
C*    IERR = error code: 0 if no error
C*                       1 if X, N, or DELTA out of range (see above).
C*                       2 if accuracy (DELTA) not achieved or in doubt.
C*-------------------
      DOUBLE PRECISION X,RES,RN,RNLES1
C...HANDLE CASE OF N=0 HERE; USE BESSP1 FOR OTHER CASES:
      IF(N.NE.0) GO TO 110
      IERR=0
      RES=DCOS(X)
      RETURN
  110 CALL BESSP1(X,N,DELTA,RN,IERR)
      IF(IERR.NE.0) PRINT 10, N,X,IERR
   10 FORMAT('0 ERROR IN SUBPROGRAM RBES1P WITH:',/,
     : '   N = ',I4,', X = ', D20.14,'; IERR =', I2)
      CALL BESSP1(X,N-1,DELTA,RNLES1,IERR)
      RES=X*RN - DFLOAT(N)*RNLES1
      RETURN
      END
C*<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>
```

```
C*
      SUBROUTINE RBES2P(X,N,RES,IERR)
C*
C*..RBES2P: SUBPROGRAM TO CALCULATE THE DERIVATIVE OF THE RICCATI-BESSEL
C*  FUNCTION OF THE 2nd KIND.
C*  THIS PROGRAM USES THE SUBPROGRAM BESSP2.
C*
C*..Reference: Stegun and Abramowitz, Eqs. 10.3.1, 10.1.21.
C*  (See SUBROUTINE BESSP2 for complete references. Note also the comments there
C*  re the recursive nature of the calculations and how to save computing time.)
C*
C*..Dr. Louis A. DeAcetis, February 1980.
C*
C*-------------------
C*
C***Form of the CALL statement:
C*
C*    CALL RBES2P(X,N,RES,IERR)
C*
C*  Input values:
C*    X = value of the argument; range:          0  < X <= 100
C*        (This must be a DOUBLE PRECISION value.)
C*    N = order; must be an integer in the range: 0 <= N <= 100
C*
C*  Output values:
C*    RES = results (derivative of the Riccati-Bessel function)
C*          (This is a DOUBLE PRECISION quantity)
C*          The mantissas of values of RES with magnitudes of
C*          less than 1.D-70 have no significance.
C*    IERR = error code: 0 if no error
C*                       1 if X or N out of range (see above).
C*-------------------
      DOUBLE PRECISION X,RES,RN,RNLES1
C...HANDLE CASE OF N=0 HERE; USE BESSP2 FOR OTHER CASES:
C...TEST X. MUST BE 0 < X <= 100.
      IF(X.GT.0.DO.AND.X.LE.1.D2) GO TO 100
      IERR=1
      RETURN
  100 IF(N.NE.0) GO TO 110
      IERR=0
      RES=DSIN(X)
      RETURN
  110 CALL BESSP2(X,N,RN,IERR)
      IF(IERR.NE.0) PRINT 10, N,X,IERR
   10 FORMAT('0 ERROR IN SUBPROGRAM RBES2P WITH:',/,
     : '   N = ',I4,', X = ', D20.14,'; IERR =', I2)
      CALL BESSP2(X,N-1,RNLES1,IERR)
      RES=X*RN - DFLOAT(N)*RNLES1
      RETURN
      END
C*<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>
```

```
C*
      SUBROUTINE BESFR1(X,N,DELTA,RES,IERR)
C*
C*..BESFR1: SUBPROGRAM TO CALCULATE THE FRACTIONAL-BESSEL FUNCTION
C*  OF THE 1st KIND. THIS PROGRAM USES THE SUBPROGRAM BESSP1.
C*
C*..Reference: Stegun and Abramowitz, Eq. 10.1.1.
C*  (See SUBROUTINE BESSP1 for complete references. Note also the comments there
C*  re the recursive nature of the calculations and how to save computing time.)
C*
C*..Dr. Louis A. DeAcetis, February 1980.
C*
C*-------------------
C*
C***Form of the CALL statement:
C*
C*    CALL BESFR1(X,N,DELTA,RES,IERR)
C*
C*  Input values:
C*    X = value of the argument; range:    0 <= X <= 100
C*        (This must be a DOUBLE PRECISION value.)
C*    N = integer part of the order. Yields the order: N + 1/2
C*        must be an integer in the range: 0 <= N <= 100
C*    DELTA = Desired accuracy expressed as the fractional error.
C*
C*  Output values:
C*    RES = results = value of the Fractional-Bessel function, J<N+.5>(X)
C*          (This is a DOUBLE PRECISION quantity)
C*          The mantissas of values of RES with magnitudes of
C*          less than 1.D-70 have no significance.
C*    IERR = error code: 0 if no error
C*                       1 if X, N, or DELTA out of range (see above).
C*                       2 if accuracy (DELTA) not achieved or in doubt.
C*-------------------
      DOUBLE PRECISION X,RES,SQR2PI
C...SQR2PI = SQRT(2/PI)
      DATA SQR2PI/.7978845608028654/
C...CASE FOR X=0, ---> RES=0.
      IERR=0
      IF(X.NE.0.D0) GO TO 100
      RES=0.D0
      RETURN
C...HANDLE CASE OF N=0 HERE. USE BESSP1 FOR OTHER CASES:
  100 IF(N.NE.0) GO TO 110
      RES = SQR2PI*DSIN(X)/DSQRT(X)
      RETURN
  110 CALL BESSP1(X,N,DELTA,RES,IERR)
      RES=SQR2PI*DSQRT(X)*RES
      RETURN
      END
C*<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>
```

```
C*
      SUBROUTINE BESFR2(X,N,RES,IERR)
C*
C*..BESFR2: SUBPROGRAM TO CALCULATE THE FRACTIONAL-BESSEL FUNCTION
C*  OF THE 2nd KIND. THIS PROGRAM USES THE SUBPROGRAM BESSP2.
C*
C*..Reference: Stegun and Abramowitz, Eq. 10.1.1.
C*  (See SUBROUTINE BESSP2 for complete references. Note also the comments there
C*  re the recursive nature of the calculations and how to save computing time.)
C*
C*..Dr. Louis A. DeAcetis, February 1980.
C*
C*-------------------
C*
C***Form of the CALL statement:
C*
C*    CALL BESFR2(X,N,RES,IERR)
C*
C*  Input values:
C*    X = value of the argument; range:    0 <= X <= 100
C*        (This must be a DOUBLE PRECISION value.)
C*    N = integer part of the order. Yields the order: N + 1/2;
C*        must be an integer in the range: 0 <= N <= 100
C*
C*  Output values:
C*    RES = results = value of the Fractional-Bessel function, Y<N+.5>(X)
C*          (This is a DOUBLE PRECISION quantity)
C*          The mantissas of values of RES with magnitudes of
C*          less than 1.D-70 have no significance.
C*    IERR = error code: 0 if no error
C*                       1 if X, N, or DELTA out of range (see above).
C*-------------------
      DOUBLE PRECISION X,RES,SQR2PI
C...SQR2PI = SQRT(2/PI)
      DATA SQR2PI/.7978845608028654/
C...CASE FOR X=0, ---> RES=-INFINITY FOR EVEN N, +INFINITY FOR ODD N
      IERR=0
      IF(X.NE.0.D0) GO TO 100
      RES=-1.D70
      IF((N/2)*2-N .NE.0) RES=-RES
      RETURN
C...HANDLE CASE OF N=0 HERE. GO TO BESSP2 FOR OTHER CASES:
  100 IF(N.NE.0) GO TO 110
      RES = -SQR2PI*DCOS(X)/DSQRT(X)
      RETURN
  110 CALL BESSP2(X,N,RES,IERR)
      RES=SQR2PI*DSQRT(X)*RES
      RETURN
      END
C*<><><><><><><><><><><><><><><><><><><><><><><><><><><><><><>
```

```
C*
      SUBROUTINE BESSP1(X,N,DELTA,RES,IERR)
C*
C*..BESSP1: SUBPROGRAM TO CALCULATE THE SPHERICAL BESSEL FUNCTION OF THE
C*  1st KIND j (x).
C*           n
C*..Dr. Louis A. DeAcetis
C*  Physics Department
C*  Bronx Community College/CUNY
C*  Bronx, NY 10453
C*
C*  LAST UPDATE: FEBRUARY 1980
C*
C*..THIS WORK WAS SUPPORTED BY A GRANT FROM THE USAF OFFICE
C*  OF SCIENTIFIC RESEARCH, #790035.
C*------------------
C*
C*..References:
C*    Abramowitz,M. and I.A. Stegun, HANDBOOK OF MATHEMATICAL FUNCTIONS,
C*    Dover publications, New york 1965, Eqs. 10.1.10, 10.1.11, 10.1.19.
C*
C*    Stegun, I.A., and M. Abramowitz, "Generation of Bessel Functions
C*    on High Speed Computers," MATHEMATICAL TABLES AND OTHER AIDS TO
C*    COMPUTATION, Vol. 11, 255-257 (1957).
C*
C*    Jahnke, E. and F. Emde, TABLES OF FUNCTIONS, Dover Publications,
C*    New York, 1945, 128.
C*
C*    Kerker, M., THE SCATTERING OF LIGHT AND OTHER ELECTROMAGNETIC
C*    RADIATION, Academic Press, New York, 1969, 65-67.
C*-------------------
C*
C***Form of the CALL statement:
C*
C*     CALL BESSP1(X,N,DELTA,RES,IERR)
C*
C*  Input values:
C*     X = value of the argument; range:          0 <= X <= 100
C*         (This must be a DOUBLE PRECISION value.)
C*     N = order; must be an integer in the range: 0 <= N <= 100
C*     DELTA = Desired accuracy expressed as the fractional error.
C*
C*  Output values:
C*     RES = results (value of the spherical Bessel function)
C*           (This is a DOUBLE PRECISION quantity)
C*           The mantissas of values of RES with magnitudes of
C*           less than 1.D-70 have no significance.
C*     IERR = error code: 0 if no error
C*                        1 if X, N, or DELTA out of range (see above).
C*                        2 if accuracy (DELTA) not achieved or in doubt.
C*-----------------
```

```
C*
C*..Because of the recursive nature of the calculations done by
C*  this SUBROUTINE, some calculated values with N>5 are stored in an
C*  array containing all of the values of the  spherical Bessel
C*  function jn(X) up to the order N that was supplied in the
C*  CALL statement. For a given value of X, if more than one order
C*  is to be used, call this SUBROUTINE first with the largest
C*  value of N to be used. Subsequent calls can then extract values
C*  from the array without recalculation, saving computing time.
C*
C*  A new calculation is initiated whenever
C*    (a) a new value of X or DELTA is supplied, or
C*    (b) N is larger than the previously used maximum value.
C*-----------------
      DOUBLE PRECISION VALUES(102),X,RES,XOLD,RESOLD
      DOUBLE PRECISION A,B,C,D,E,F,T1,T2,DS,DC
      DATA C,XOLD,NOLD/0.,0.,0/
      IERR=0
      RES=0.D0
      IF(N.LT.0.OR.N.GT.100.OR.X.LT.0..OR.X.GT.1.D2)IERR=1
      IF(DELTA.LT.1.E-13) IERR=1
      IF(IERR.NE.0) RETURN
C...HANDLE CASE WHERE X=0 AND N>0: RES=0
      RES=0.D0
      IF(X.EQ.0..AND.N.NE.0) RETURN
C...X=0, N=0: RES=1;   X>0, N=0: RES=SIN(X)/X
      RES=1.D0
      IF(X.EQ.0..AND.N.EQ.0) RETURN
      IF(N.GT.0) GO TO 100
      RES=DSIN(X)/X
      RETURN
C...FOLLOWING FOR X>0, N>0 CASE:
  100 IF((X.EQ.XOLD.AND.N.LT.NOLD.AND.NOLD.GT.4).AND.
     : (C.NE.1.D0.OR.N.GT.4)) GO TO 400
      DS=DSIN(X)
      DC=DCOS(X)
      IF(N.GT.4) GOTO 200
      GO TO (110,120,130,140),N
C...N=4:
  140 RES=(((105.D0/(X*X)-45.D0)/X+X)*DS-(105.D0/(X*X)-10.D0)*DC)/X/X
      RETURN
C...N=3:
  130 RES=((15.D0/X/X-6.D0)*DS-(15.D0/X-X)*DC)/(X*X)
      RETURN
C...N=2:
  120 RES=((3.D0/X-X)*DS-3.D0*DC)/(X*X)
      RETURN
C...N=1:
  110 RES=(DS/X-DC)/X
      RETURN
```

```
C...USING PREVIOUSLY GENERATED VALUES WHEN X UNCHANGED AND N<LAST N
  400 RES=C*VALUES(N+1)
      RETURN
  101 RES=VALUES(N+1)
      RETURN
C...X>0, N>4:
  200 JK=4
      A=(15.D0/(X*X)-6.D0)/(X*X)
      B=((105.D0/(X*X)-45.D0)/X+X)/(X*X)
      E=(105.D0/(X*X)-10.D0)/(X*X)
      F=(X-15.D0/X)/(X*X)
  160 C=(2*JK+1)*B/X - A
      JK=JK+1
      D=(1-2*JK)*E/X - F
      T1=C*DS
      T2=D*DC
      IF(2*(JK/2)-JK.EQ.0) T2=-T2
C...IF THIS METHOD BECOMES TOO INACCURATE, GO TO ALTERNATE.
      IF(T1*T2.LT.0..AND.ABS(T1+T2).LT.1.D-14/DELTA*ABS(T1)) GO TO 500
      VALUES(JK+1)=T1+T2
      IF(JK.EQ.N) GO TO 214
      A=B
      B=C
      F=E
      E=D
      GO TO 160
C...ALTERNATE METHOD-- "MILLER'S METHOD"
C   FIND INITIAL VALUE FOR JM WHICH IS LARGER THAN N OR X BUT <= 102
  500 A=DS/X
      RESOLD=0.D0
      JM=MAX(N+1,DINT(X+.5))
  170 JM=MIN(JM,101)
      JM=JM+1
      VALUES(JM)=0.D0
      VALUES(JM-1)=1.D0
      DO 555 K=3,JM
      KK=JM+1-K
  555 VALUES(KK)=(2.D0*DFLOAT(KK-1)+3.D0)*VALUES(KK+1)/X-VALUES(KK+2)
      C=A/VALUES(1)
      RES=C*VALUES(N+1)
      IF((DABS((RES-RESOLD)/RES).LT.DELTA).OR.(DABS(RES).LT.1D-70))
     : GO TO 215
      RESOLD=RES
      IF(JM.GT.101) GO TO 220
      JM=JM+4
      GO TO 170
C...ACCURACY IN DOUBT OR NOT ACHIEVED--
  220 IERR=2
      RETURN
  214 RES=VALUES(N+1)
      C=1.D0
```

```
C...SAVE OLD VALUES FOR LATER USE:
  215 XOLD=X
      NOLD=N
      RETURN
      END
C*
C*=============================================================
```

```
C*
      SUBROUTINE BESSP2(X,N,RES,IERR)
C*
C*..BESSP2: SUBPROGRAM TO CALCULATE THE SPHERICAL BESSEL FUNCTION OF THE
C*  2nd KIND, y (x).
C*            n
C*..Dr. Louis A. DeAcetis
C*  Physics Department
C*  Bronx Community College/CUNY
C*  Bronx, NY 10453
C*
C*  LAST UPDATE: FEBRUARY 1980
C*
C*..THIS WORK WAS SUPPORTED BY A GRANT FROM THE USAF OFFICE
C*  OF SCIENTIFIC RESEARCH, #790035.
C*-----------------
C*
C*..References:
C*   Abramowitz,M. and I.A. Stegun, HANDBOOK OF MATHEMATICAL FUNCTIONS,
C*   Dover publications, New york 1965, Eqs. 10.1.12, 10.1.19.
C*
C*   Stegun, I.A., and M. Abramowitz, "Generation of Bessel Functions
C*   on High Speed Computers," MATHEMATICAL TABLES AND OTHER AIDS TO
C*   COMPUTATION, Vol. 11, 255-257 (1957).
C*
C*   Jahnke, E. and F. Emde, TABLES OF FUNCTIONS, Dover Publications,
C*   New York, 1945, 131-132.
C*
C*   Kerker, M., THE SCATTERING OF LIGHT AND OTHER ELECTROMAGNETIC
C*   RADIATION, Academic Press, New York, 1969, 65-67.
C*-------------------
C*
C*...Form of the CALL statement:
C*
C*     CALL BESSP2(X,N,RES,IERR)
C*
C*   Input values:
C*     X = value of the argument; range:        . 0 <= X <= 100
C*         (This must be a DOUBLE PRECISION value.)
C*     N = order; must be an integer in the range: 0 <= N <= 100
C*
C*   Output values:
C*     RES = results (value of the spherical Bessel function)
C*           (This is a DOUBLE PRECISION quantity. Tests indicate
C*           that the results are good to better than seven significant
C*           digits, except for magnitudes > 1.D70.)
C*     IERR = error code: 0 if no error
C*                        1 if X or N out of range (see above).
C*-----------------
```

```
C*
C***Because  of the recursive nature of the calculations, all orders
C*  of the function will be calculated up to order N. If more than
C*  one order is to be used with the same value of X, then an initial
C*  CALL to the SUBROUTINE with the highest value of N will cause all
C*  lower orders to be calculated, and these will be used as long as
C*  X is unchanged and N < highest value. This will save some computing time.
C*
C*------------------
      DOUBLE PRECISION VALUES(102),X,RES,XOLD
      DOUBLE PRECISION DS,DC
      DATA XOLD,NOLD/0.,0/
      IERR=1
      RES=0.D0
      IF(N.LT.0.OR.N.GT.100.OR.X.LT.0..OR.X.GT.1.D2)RETURN
      IERR=0
C...HANDLE CASE WHERE X=0: RES= -INFINITY=-1.D70
      RES=-1.D70
      IF(X.EQ.0.D0) RETURN
      DC=DCOS(X)
      IF(N.EQ.0) GO TO 105
C...FOLLOWING FOR X>0, N>0 CASES:
  100 IF(X.EQ.XOLD.AND.N.LT.NOLD) GO TO 400
      DS=DSIN(X)
      IF(N.EQ.1) GO TO 110
C...N=2:
  120 VALUES(3)=((X-3.D0/X)*DC-3.D0*DS)/(X*X)
C...N=1:
  110 VALUES(2)=-(DC/X+DS)/X
C...N=0:
  105 VALUES(1)=-DC/X
      IF(N.GT.2) GO TO 200
      GO TO 250
C...USING PREVIOUSLY GENERATED VALUES WHEN X UNCHANGED AND N<HIGHEST N
  400 RES=VALUES(N+1)
      RETURN
C...USING RECURSIVE RELATIONSHIP
  200 DO 666 K=3,N
  666 VALUES(K+1)=(2.D0*DFLOAT(K)-1.D0)*VALUES(K)/X - VALUES(K-1)
  250 RES=VALUES(N+1)
      XOLD=X
      NOLD=N
      RETURN
      END
```

```
10000 REM SUBPROGRAM TO CALCULATE RICCATI/SPHERICAL BESSEL FUNCTIONS
10002 REM   OF THE 1ST KIND,AND THEIR DERIVATIVE.
10010 REM...DR. LOUIS A. DEACETIS
10011 REM   PHYSICS DEPARTMENT
10012 REM   BRONX COMMUNITY COLLEGE/CUNY
10013 REM   BRONX, NY 10453
10014 REM
10015 REM...LAST UPDATE: FEBRUARY 1980
10016 REM
10018 REM...THIS WORK WAS SUPPORTED BY A GRANT FROM THE USAF OFFICE OF
10019 REM   SCIENTIFIC RESEARCH, #790035.
10020 REM
10021 REM...REFERENCES:
10022 REM HANDBOOK OF MATHEMATICAL FUNCTIONS, ABRAMOWITZ, M. AND I.A. STEGUN,
10023 REM DOVER PUBLIC., N.Y.1965, EQS.10.1.10,10.1.11,10.1.19,10.1.21,10.3.1.
10024 REM
10025 REM STEGUN, I. A., AND M. ABRAMOWITZ, "GENERATION OF BESSEL FUNCTIONS
10026 REM FUNCTIONS ON HIGH SPEED COMPUTERS," MATHEMATICAL TABLES AND
10027 REM OTHER AIDS TO COMPUTATION, VOL. 11, 255-257 (1957).
10029 REM
10030 REM JAHNKE, E. AND F. EMDE, TABLES OF FUNCTIONS, DOVER PUBLICATIONS,
10031 REM NEW YORK, 1945, 128.
10032 REM
10033 REM KERKER, M., THE SCATTERING OF LIGHT AND OTHER ELECTROMAGNETIC
10034 REM RADIATION, ACADEMIC PRESS, NEW YORK 1969, PP. 64-67.
10036 REM ------------
10038 REM ALL VARIABLES AND FUNCTIONS USED BY/IN THESE ROUTINES BEGIN WITH J.
10040 REM   DO NOT USE VARIABLE NAMES BEGINNING WITH J ELSEWHERE.
10041 REM   THE NAMES PI, P2I, HALFPI ARE ALSO USED (= PI, 2*PI, PI/2)
10042 REM ------------
10044 REM FIRST CALL TO THESE SUBPROGRAMS IS "GOSUB 10000" WHICH INITIALIZES
10046 REM   ALL FUNCTIONS AND DEFINES CONSTANTS.
10047 REM
10048 REM TO GENERATE VALUES OF RICCATI-BESSEL FUNCTIONS, GOSUB 10100
10049 REM TO GET VALUES OF DERIVATIVES, GOSUB 10200
10050 REM TO GET VALUES OF SPHERICAL BESSEL FUNCTIONS, GOSUB 10300.
10051 REM INPUT VALUES:
10052 REM   JX = ARGUMENT (0 <= JX <= 100)
10053 REM   JN = ORDER    (0 <= JN <= 100)
10054 REM   DELTA = DESIRED ACCURACY EXPRESSED AS THE FRACTIONAL ERROR.
10055 REM OUTPUT VALUES:
10056 REM   JS = VALUE OF SPHERICAL BESSEL FUNCTION, J<JN>(JX)
10057 REM   JR = VALUE OF RICCATI FUNCTION OR ITS DERIVATIVE.
10058 REM        (MANTISSAS OF VALUES < 1E-70 HAVE NO SIGNIFICANCE)
10059 REM   JJ = ERROR CODE = 0 (NO ERROR), = 1 (JN, JX, DELTA OUT OF RANGE),
10060 REM                   = 2 (ACCURACY NOT ACHIEVED OR IN DOUBT)
10061 REM ------------
```

```
10062 REM   BECAUSE OF THE RECURSIVE NATURE OF THE CALCULATIONS, FOR VALUES
10064 REM   OF JN LARGER THAN 4, ALL ORDERS OF THE FUNCTION WILL BE CALCULATED
10066 REM   UP TO ORDER JN . THUS, IF MORE THAN ONE ORDER WITH THE SAME JX
10070 REM   IS TO BE USED, AN INITIAL GOSUB WITH THE HIGHEST VALUE OF
10072 REM   OF JN TO BE USED WILL GENERATE LOWER ORDERS AS WELL, AND
10074 REM   THESE WILL BE USED AS LONG AS JX AND DELTA ARE UNCHANGED AND
10076 REM   JN<HIGHEST VALUE (THIS CAN SAVE CONSIDERABLE COMPUTING TIME).
10080 REM ------------
10081 REM ROUTINES FOR SINE AND COSINE TO 15 DECIMAL PLACES ARE AT LINES
10082 REM   10900 - 10990. INPUT ANGLE IS AA.
10084 REM   GOSUB 10900 FOR SIN(AA) RETURNED AS VARIABLE SA.
10086 REM   GOSUB 10950 FOR COS(AA) RETURNED AS VARIABLE CA.
10088 REM  IF THE BASIC BEING USED HAS DOUBLE PRECISION FUNCTIONS, THEN OMIT
10089 REM   THESE LINES, AND CHANGE "SA" AND "CA" TO SIN AND COS BELOW.
10090 REM ------------
10091 DIM JF(101)
10092 PI=3.141592653589793: P2I=6.283185307179586: HALFPI=1.570796326794896
10094 DEF FNA(A)=A-INT(A/P2I)*P2I :REM MAKE "A" A VALUE BETWEEN 0 AND 2 PI
10096 DEF FNT(N,X)=(2*N+3)*JF(N+1)/X-JF(N+2)
10098 JC=0: JZ=0: JY=0: RETURN
10100 REM FIND THE RICCATI-BESSEL FUNCTION OF THE FIRST KIND:
10102 IF JN<0 OR JX<0 OR DELTA<1.E-13  THEN JR=0: JJ=1: RETURN
10106 JJ=0
10108 IF JX=0 THEN JR=0: RETURN:  REM JR=0 IF JX=0
10110 IF JN<>0 GOTO 10120
10115 AA=JX: GOSUB 10900: JR=SA: RETURN
10120 GOSUB 10300
10130 JR=JX*JS: RETURN
10107 REM HANDLE JX,JN= 0 CASES HERE: USE SPHERICAL BESSEL ROUTINE FOR OTHERS:
10200 REM FIND FIRST DERIVATIVE OF THE RICCATI-BESSEL FUNCTION OF FIRST KIND:
10201 IF JN<0ORJX<0 OR DELTA<1.E-13 THEN JR=0: JJ=1: RETURN
10202 JJ=0: IF JN<>0 GO TO 10206
10204 AA=JX: GOSUB 10950: JR=CA: RETURN
10205 JJ=0
10206 GOSUB 10300: JT=JS: IF JJ<>0 PRINT 'ERROR FOR JN,JX= ';JN,JX
10208 JN=JN-1: GOSUB 10300: JN=JN+1
10209 JR=JX*JT-JN*JS: RETURN
10300 REM FIND SPHERICAL BESSEL FUNCTION OF THE FIRST KIND:
10301 IF JN<0 OR JX<0 OR DELTA<1.E-13 THEN JS=0: JJ=1: RETURN
10305 IF JX=0ANDJN<>0 THEN JS=0:RETURN:REM JS=0 FOR JX=0 EXCEPT IF JN=0
10311 IF JX=JY AND JN<=JZ AND JZ>4 AND (JC<>1 OR JN>4) GOTO 10400
10312 AA=JX: GOSUB 10900: IF JN<>0 GOTO 10316
10314 IF JX=0 THEN JS=1 ELSE JS=SA/JX
10315 RETURN
10316 GOSUB 10950: ON JN GOTO 10320,10330,10340,10350
10318 GOTO 10360: REM JN>=5
10320 JS=(SA/JX - CA)/JX: RETURN : REM JN=1
10330 JS=((3/JX - JX)*SA - 3*CA)/(JX*JX): RETURN: REM JN=2
10340 JS=((15/(JX*JX)-6)*SA-(15/JX-JX)*CA)/(JX*JX):RETURN: REM JN=3
10350 JS=(((105/(JX*JX)-45)/JX+JX)*SA-(105/(JX*JX)-10)*CA)/(JX*JX)
10352 RETURN :REM JN=4
```

```
10360 JK=4: JA=(15/(JX*JX)-6)/(JX*JX) : JB=((105/(JX*JX)-45)/JX+JX)/(JX*JX)
10370 JE=(105/(JX*JX)-10)/(JX*JX) : JF=(JX-15/JX)/(JX*JX)
10380 JC=(2*JK+1)*JB/JX - JA
10385 JK=JK+1
10390 JD=(1-2*JK)*JE/JX-JF
10394 T1=JC*SA: T2=JD*CA: IF(2*INT(JK/2)-JK)=0 THEN T2=-T2: REM (-1)|(N+1)
10396 IF T1*T2<0 AND ABS(T1+T2)<1.E-14/DELTA*ABS(T1) GOTO 10500
10397 JF(JK)=T1+T2
10398 IF JK=JN THEN JS=JF(JN): JC=1: GOTO 10600
10399 JA=JB: JB=JC: JF=JE: JE=JD: GOTO 10380
10400 REM USING PREVIOUSLY GENERATED VALUES WHEN JX UNCHANGED AND JN<OLD JN
10410 JS=JC*JF(JN): RETURN
10500 PRINT "USING MILLER'S METHOD"
10502 JA=SA/JX :REM FIND LARGER VALUE OF JX,JN <100:
10505 JF=0:JM=INT(-JX*(JX>JN)-JN*(JN>JX)+.5):REM CHOOSE JM=MAX(JX,JN)+1<=100
10508 JM=-JM*(JM<101)-100*(100<JM): JM=JM+1
10510 JF(JM)=0: JF(JM-1)=1
10520 FOR JK=JM-2 TO 0 STEP -1
10530 JF(JK)=FNT(JK,JX): NEXT JK
10550 JC=JA/JF(0): REM JC= SCALING CONSTANT
10555 JS=JC*JF(JN)
10571 REM FINISHED IF VALUE < 1E-70 OR DESIRED ACCURACY ACHIEVED
10572 IF(ABS((JS-JF)/JS)<DELTA) OR (ABS(JS)<1.E-70) GOTO 10600
10574 JF=JS: IF JM<100 THEN JM=JM+5: GOTO 10508
10576 JJ=2
10580 RETURN
10600 JZ=JN: JY=JX: RETURN: REM SAVE OLD VALUES FOR USE IN 10400
10900 REM......**SERIES FOR SIN & COS**.......
10905 REM    INPUT ANGLE=AA. OUTPUT: SA=SIN(AA); CA=COS(AA)
10907 REM..VARIABLE NAMES USED: AA, CA, SA, SN, XX CONSTANTS: PI, HALFPI
10908 REM    FUNCTION: FNA(X)
10910 REM...SINE(AA):
10915 IF AA=0 THEN SA=0: RETURN
10920 XX=FNA(AA): SN=1: IF XX>PI THEN SN=-1: XX=XX-PI
10925 IF XX>HALFPI THEN XX=PI-XX
10930 SA=1-XX*XX/110*(1-XX*XX/156*(1-XX*XX/210*(1-XX*XX/272*(1-XX*XX/342))))
10935 SA=XX*(1-XX*XX/6*(1-XX*XX/20*(1-XX*XX/42*(1-XX*XX/72*SA))))
10940 SA=SN*SA: RETURN
10950 REM..COSINE(AA):
10955 IF AA=0 THEN CA=1: RETURN
10960 XX=FNA(AA): SN=1: IF XX>PI THEN XX=XX-PI: SN=-1
10965 IF XX>HALFPI THEN XX=PI-XX: SN=-SN
10970 CA=1-XX*XX/132*(1-XX*XX/182*(1-XX*XX/240*(1-XX*XX/306*(1-XX*XX/380))))
10975 CA=1-XX*XX/2*(1-XX*XX/12*(1-XX*XX/30*(1-XX*XX/56*(1-XX*XX/90*CA))))
10980 CA=SN*CA: RETURN
10990 REM.......**END OF SIN & COS ROUTINES**.......
```

```
10000 REM SUBPROGRAM TO CALCULATE RICCATI/SPHERICAL BESSEL FUNCTIONS
10002 REM   OF THE 2ND KIND AND THEIR DERIVATIVE.
10010 REM...DR. LOUIS A DEACETIS
10011 REM   PHYSICS DEPARTMENT
10012 REM   BRONX COMMUNITY COLLEGE/CUNY
10013 REM   BRONX, NY 10453
10014 REM
10015 REM..LAST UPDATE: FEBRUARY 1980
10016 REM
10017 REM...THIS WORK WAS SUPPORTED BY A GRANT FROM THE USAF OFFICE OF
10018 REM   SCIENTIFIC RESEARCH, #790035
10019 REM ------------
10021 REM..REFERENCES:
10022 REM HANDBOOK OF MATHEMATICAL FUNCTIONS, ABRAMOWITZ, M. AND I.A. STEGUN,
10023 REM DOVER PUBLICATIONS, NEW YORK 1965, EQS.10.1.12,10.1.19,10.3.1,10.1.21
10024 REM
10025 REM STEGUN, I.A., AND M ABRAMOWITZ, "GENERATION OF BESSEL FUNCTIONS
10026 REM FUNCTIONS ON HIGH SPEED COMPUTERS," MATHEMATICAL TABLES AND
10027 REM OTHER AIDS TO COMPUTATION, VOL. 11, 255-257 (1957).
10029 REM
10030 REM JAHNKE, E. AND F. EMDE, TABLES OF FUNCTIONS, DOVER PUBLICATIONS,
10031 REM NEW YORK, 1945, 126-129.
10032 REM
10033 REM KERKER, M., THE SCATTERING OF LIGHT AND OTHER ELECTROMAGNETIC
10034 REM RADIATION, ACADEMIC PRESS, NEW YORK 1969, PP. 64-67.
10036 REM ------------
10038 REM ALL VARIABLES AND FUNCTIONS USED BY/IN THESE ROUTINES BEGIN WITH Y.
10040 REM   DO NOT USE VARIABLE NAMES BEGINNING WITH Y ELSEWHERE.
10041 REM   THE NAMES PI, P2I AND HALFPI ARE ALSO USED (= PI, 2*PI, PI/2)
10042 REM ------------
10044 REM FIRST CALL TO THESE SUBPROGRAMS IS "GOSUB 10000" WHICH DEFINES
10046 REM   ALL FUNCTIONS AND CONSTANTS, AND INITIALIZES VARIABLES
10047 REM
10048 REM TO GENERATE VALUES OF THE Y-RICCATI-BESSEL FUNCTIONS, GOSUB 10100
10049 REM TO GET VALUES OF THE DERIVATIVE OF THE Y-RICCATI'S, GOSUB 10200
10050 REM TO GET THE SPHERICAL BESSEL FUNCTIONS OF THE 2ND KIND, GOSUB 10300
10051 REM INPUT VALUES:
10052 REM   YX = ARGUMENT (0 <= YX <= 100)  (0 <YX <= 100 FOR DERIVATIVE)
10053 REM   YN = ORDER    (0 <= YN <= 100)
10054 REM
10055 REM OUTPUT VALUES:
10056 REM   YS = VALUE OF SPHERICAL BESSEL FUNCTION Y<YN>(YX)
10057 REM   YR = VALUE OF THE RICCATI-BESSEL FUNCTION OR ITS DERIVATIVE
10058 REM        (ACCURACY FOR YR AND YS BETTER THAN 5 DIGITS)
10059 REM   JJ = ERROR CODE = 0 (NO ERROR), = 1 (YN, YX OUT OF RANGE)
10060 REM ------------
```

```
10062 REM   BECAUSE OF THE RECURSIVE NATURE OF THE CALCULATIONS, ALL ORDERS
10064 REM   OF THE FUNCTION WILL BE CALCULATED UP TO ORDER YN IF MORE THAN
10066 REM   ONE ORDER IS TO BE USED WITH THE SAME VALUE OF YX, THEN AN INITIAL
10072 REM   GOSUB WITH THE HIGHEST VALUE OF YN WILL CAUSE ALL LOWER
10074 REM   ORDERS TO BE CALCULATED AND THESE WILL BE USED AS LONG AS YX
10076 REM   IS UNCHANGED AND YN<HIGHEST VALUE (THIS CAN SAVE CONSIDERABLE
10078 REM   COMPUTING TIME).
10080 REM ------------
10081 REM ROUTINES FOR SINE AND COSINE TO 15 DECIMAL PLACES ARE AT LINES
10082 REM   10900 - 10990. INPUT ANGLE IS AA.
10084 REM   GOSUB 10900 FOR SIN(AA) RETURNED AS VARIABLE SA.
10086 REM   GOSUB 10950 FOR COS(AA) RETURNED AS VARIABLE CA.
10088 REM  IF THE BASIC BEING USED HAS DOUBLE PRECISION FUNCTIONS, THEN OMIT
10089 REM   THESE LINES, AND CHANGE "SA" AND "CA" TO SIN AND COS BELOW.
10090 REM ------------
10091 DIM YF(101)
10092 PI=3.141592653589793: P2I=6.283185307179586: HALFPI=1.570796326794896
10094 DEF FNA(A)=A-INT(A/P2I)*P2I :REM MAKE "A" A VALUE BETWEEN 0 AND 2 PI
10096 DEF FNT(N,X)=(2*N-1)*YF(N-1)/X-YF(N-2)
10098 YZ=0: YY=0: RETURN
10100 REM FIND RICCATTI-BESSEL FUNCTION OF THE SECOND KIND:
10103 IF YN<0 OR YX<0 THEN YR=0: JJ=1: RETURN
10105 JJ=0
10108 IF YX=0ANDYN<>0THEN YR=-1.E70:RETURN:REM YR=-INFINITY FOR YX=0 AND YN<>0
10110 IF YX=0 AND YN=0 THEN YR=-1: RETURN: REM YR=-1 FOR YX=0 AND YN=0
10120 GOSUB 10311: YR=YX*YS: RETURN
10200 REM FIND FIRST DERIVATIVE OF RICCATI-BESSEL FUNCTION OF THE SECOND KIND:
10210 IF YX>0 AND YX<=100 GOTO 10220
10212 JJ=1: RETURN
10220 JJ=0: IF YN<> GOTO 10230
10222 AA=YX: GOSUB 10900: YR=SA: RETURN: REM YN=0 ==> DERIV=SIN(YX)
10230 GOSUB 10300:YT=YS:IF JJ<>0 THEN PRINT'ERROR IN DERIV: YN,YX= ';YN,YX
10232 YN=YN-1: GOSUB 10300: YN=YN+1
10234 YR=YX*YT-YN*YS: RETURN
10300 REM FIND SPHERICAL BESSEL FUNCTION OF THE SECOND KIND:
10302 IF YN<0 OR YX<0 THEN YS=0: JJ=1: RETURN
10305 JJ=0: IF YX=0 THEN YS=-1.E70: RETURN: REM YS=-INFINITY FOR YX=0
10311 IF YX=YY AND YN<=YZ GOTO 10400
10312 AA=YX: GOSUB 10950: IF YN=0 GOTO 10345
10316 GOSUB 10900: ON YN GOTO 10340,10330
10330 YF(2)=((YX - 3/YX)*CA - 3*SA)/(YX*YX): REM YN=2
10340 YF(1)=-(CA/YX + SA)/YX: REM YN=1
10345 YF(0)=-CA/YX: REM YN=0
10350 IF YN>2 GOTO 10500
10360 GOTO 10540
```

```
10400 REM USING PREVIOUSLY GENERATED VALUES WHEN YX UNCHANGED AND YN<OLD YN
10410 YS=YF(YN): RETURN
10500 PRINT 'USING RECURSION RELATIONSHIP'
10510 FOR YK=3 TO YN
10520 YF(YK)=FNT(YK,YX)
10530 NEXT YK
10540 YS=YF(YN)
10600 YZ=YN: YY=YX: RETURN: REM SAVE OLD VALUES FOR USE IN 10400
10900 REM.......**SERIES FOR SIN & COS**........
10905 REM   INPUT ANGLE=AA. OUTPUT: SA=SIN(AA); CA=COS(AA)
10907 REM...VARIABLE NAMES USED: AA, CA, SA, SN, XX. CONSTANTS: PI, HALFPI
10908 REM   FUNCTION: FNA(X)
10910 REM...SINE(AA):
10915 IF AA=0 THEN SA=0: RETURN
10920 XX=FNA(AA): SN=1: IF XX>PI THEN SN=-1: XX=XX-PI
10925 IF XX>HALFPI THEN XX=PI-XX
10930 SA=1-XX*XX/110*(1-XX*XX/156*(1-XX*XX/210*(1-XX*XX/272*(1-XX*XX/342))))
10935 SA=XX*(1-XX*XX/6*(1-XX*XX/20*(1-XX*XX/42*(1-XX*XX/72*SA))))
10940 SA=SN*SA: RETURN
10950 REM...COSINE(AA):
10955 IF AA=0 THEN CA=1: RETURN
10960 XX=FNA(AA): SN=1: IF XX>PI THEN XX=XX-PI: SN=-1
10965 IF XX>HALFPI THEN XX=PI-XX: SN=-SN
10970 CA=1-XX*XX/132*(1-XX*XX/182*(1-XX*XX/240*(1-XX*XX/306*(1-XX*XX/380))))
10975 CA=1-XX*XX/2*(1-XX*XX/12*(1-XX*XX/30*(1-XX*XX/56*(1-XX*XX/90*CA))))
10980 CA=SN*CA: RETURN
10990 REM........**END OF SIN & COS ROUTINES**.......
```